

On p -almost direct products and residual properties of pure braid groups of nonorientable surfaces

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Abstract

We prove that the n^{th} pure braid group of a nonorientable surface (closed or with boundary, but different from \mathbb{RP}^2) is residually 2-finite. Consequently, this group is residually nilpotent. The key ingredient in the closed case is the notion of p -almost direct product, which is a generalization of the notion of almost direct product. We prove therefore also some results on lower central series and augmentation ideals of p -almost direct products.

1 Introduction

Let M be a surface (orientable or not, possibly with boundary) and $F_n(M) = \{(x_1, \dots, x_n) \in M^n / x_i \neq x_j \text{ for } i \neq j\}$ its n^{th} configuration space. The fundamental group $\pi_1(F_n(M))$ is called the n^{th} *pure braid group* of M and shall be denoted by $P_n(M)$.

The mapping class group $\Gamma(M)$ of M is the group of isotopy classes of homeomorphisms $h : M \rightarrow M$ which are identity on the boundary. Let $\mathcal{X}_n = \{z_1, \dots, z_n\}$ a set of n distinguished points in the interior of M ; the pure mapping class group $\text{P}\Gamma(M, \mathcal{X}_n)$ relatively to \mathcal{X}_n is the group of the isotopy classes of homeomorphisms $h : M \rightarrow M$ satisfying $h(z_i) = z_i$ for all i : since this group does not depend on the choice of the set \mathcal{X}_n but only on its cardinality we can write $\text{P}_n\Gamma(M)$ instead of $\text{P}\Gamma(M, \mathcal{X}_n)$. Forgetting the marked points, we get a morphism $\text{P}_n\Gamma(M) \rightarrow \Gamma(M)$ whose kernel is known to be isomorphic to $P_n(M)$ when M is not a sphere, a torus, a projective plane or a Klein bottle (see [Sc, GJ]).

Now, recall that if \mathcal{P} is a group-theoretic property, then a group G is said to be *residually* \mathcal{P} if, for all $g \in G$, $g \neq 1$, there exists a group homomorphism $\varphi : G \rightarrow H$ such that H is \mathcal{P} and $\varphi(g) \neq 1$. We are interested in this paper to the following properties: nilpotence, being free and being a finite p -group for a prime number p (mostly $p = 2$). Recall that, if for subgroups H and K of G , $[H, K]$ is the subgroup generated by $\{[h, k] / (h, k) \in H \times K\}$ where $[h, k] = h^{-1}k^{-1}hk$, the lower central series of G , $(\Gamma_k G)_{k \geq 1}$, is defined inductively by $\Gamma_1 G = G$ and $\Gamma_{k+1} G = [G, \Gamma_k G]$. It is well known that G is residually nilpotent if, and only if, $\bigcap_{k=1}^{+\infty} \Gamma_k G = \{1\}$. From the lower central series of G one can define another filtration $D_1(G) \supseteq D_2(G) \supseteq \dots$ setting $D_1(G) = G$, and for $i \geq 2$, defining $D_i(G) = \{x \in G / \exists n \in \mathbb{N}^*, x^n \in \Gamma_i(G)\}$. After Garoufalidis and Levine [GLe], this filtration is called *rational lower central series* of G and a group G is residually torsion-free nilpotent if, and only if, $\bigcap_{i=1}^{\infty} D_i(G) = \{1\}$.

When M is an orientable surface of positive genus (possibly with boundary) or a disc with holes, it is proved in [BGG] and [BB] that $P_n(M)$ is residually torsion-free nilpotent for all $n \geq 1$. The fact that a

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group is residually torsion-free nilpotent has several important consequences, notably that the group is bi-orderable [MR] and residually p -finite [Gr]. The goal of this article is to study the nonorientable case and, more precisely, to prove the following:

THEOREM 1 *The n^{th} pure braid group of a nonorientable surface different from \mathbb{RP}^2 is residually 2-finite.*

In the case of $P_n(\mathbb{RP}^2)$ we give some partial results at the end of Section 4. Since a finite 2-group is nilpotent, a residually 2-finite group is residually nilpotent. Thus, we have

COROLLARY 1 *The n^{th} pure braid group of a nonorientable surface different from \mathbb{RP}^2 is residually nilpotent.*

Remark that in [Go] it was shown that the n^{th} pure braid group of a nonorientable surface is not bi-orderable and therefore it is not residually torsion-free nilpotent. Let us notice also that if pure braid groups of nonorientable surfaces with boundary are residually p for a prime $p \neq 2$ therefore pure braid groups of nonorientable closed surfaces are also residually p (Remark 3); however since the technique proposed in the nonorientable case applies only for $p = 2$, the question if pure braid groups of nonorientable surfaces are residually p for $p \neq 2$ is still open (recall that there are groups residually p for infinitely many primes p which are not residually torsion-free nilpotent, see [H]).

Remark that one can prove that finite type invariants separate classical braids using the fact that the pure braid group P_n is residually nilpotent without torsion (see [Pa]). Moreover using above residual properties it is possible to construct algebraically a universal finite type invariant over \mathbb{Z} on the classical braid group B_n (see [Pa]). Similar constructions were afterwards proposed for braids on orientable surfaces (see [BF, GP]): in a further paper we will explore the relevance of Theorem 1 in the realm of finite type invariants over $\mathbb{Z}/2\mathbb{Z}$ for braids on non orientable surfaces.

From now on, $M = N_{g,b}$ is a nonorientable surface of genus g with b boundary components, simply denoted by N_g when $b = 0$. We will see N_g as a sphere S^2 with g open discs removed and g Möbius strips glued on each circle (see figure 2 where each crossed disc represents a Möbius strip). The surface $N_{g,b}$ is obtained from N_g by removing b open discs. The mapping class groups $\Gamma(N_{g,b})$ and pure mapping class group $P_n\Gamma(N_{g,b})$ will be denoted respectively $\Gamma_{g,b}$ and $\Gamma_{g,b}^n$.

The paper is organized as follows. In Section 2, we prove Theorem 1 for surfaces with boundary: following what the authors did in the orientable case (see [BGG]), we embed $P_n(N_{g,b})$ in a Torelli group. The difference here is that we must consider mod 2 Torelli groups. In Section 3 we introduce the notion of p -almost direct product, which generalizes the notion of almost direct product (see Definition 1) and we prove some results on lower central series and augmentations ideals of p -almost direct products (Theorems 4 and 5) that can be compared with similar results on almost direct products (Theorem 3.1 in [FR] and Theorem 3.1 in [Pa]).

In Section 4, the existence of a split exact sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \longrightarrow \pi_1(N_g) \longrightarrow 1$$

and results from Section 2 and 3 are used to prove Theorem 1 in the closed case (Theorem 7). The method is similar to the one developed for orientable surface in [BB]: the difference will be that the semi-direct product $P_{n-1}(N_{g,1}) \rtimes \pi_1(N_g)$ is a 2-almost-direct product (and not an almost-direct product as in the case of closed oriented surfaces).

For proving the main result of the paper, we will also need a group presentation for $P_n(N_{g,b})$ when $b \geq 1$. Although generators of this group seem to be known, we could not find a group presentation in the literature. Thus, we give one in the Appendix (Theorem A).

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2 The case of non-empty boundary

In this section, $N = N_{g,b}$ is a nonorientable surface of genus $g \geq 1$ with boundary (ie $b \geq 1$). In this case, one has $P_n(N) = \text{Ker}(\Gamma_{g,b}^n \longrightarrow \Gamma_{g,b})$ for all $n \geq 1$.

2.1 Notations

We will follow notations from [PS]. A simple closed curve in N is an embedding $\alpha : S^1 \longrightarrow N \setminus \partial N$. Such a curve is said two-sided (resp. one-sided) if it admits a regular neighborhood homeomorphic to an annulus (resp. a Möbius strip). We shall consider the following elements in $\Gamma_{g,b}$.

- If α is a two-sided simple closed curve in N , τ_α is a Dehn twist along α .
- Let μ and α be two simple closed curves such that μ is one-sided, α is two-sided and $|\alpha \cap \mu| = 1$. A regular neighborhood K (resp. M) of $\alpha \cup \mu$ (resp. μ) is diffeomorphic to a Klein bottle with one hole (resp. a Möbius strip). Pushing M once along α , we get a diffeomorphism of K fixing the boundary: it can be extended via the identity to N . Such a diffeomorphism is called a crosscap slide, and denoted by $Y_{\mu,\alpha}$.
- Consider a one-sided simple closed curve μ containing exactly one marked point z_i . Sliding z_i once along μ , we get a diffeomorphism S_μ of N which is identity outside a regular neighborhood of μ . Such a diffeomorphism will be called puncture slide along μ .

2.2 Blowup homomorphism

In this section, we recall the construction of the Blowup homomorphism $\eta_{g,b}^n : \Gamma_{g,b}^n \longrightarrow \Gamma_{g+n,b}$ given in [Sz1], [Sz2] and [PS].

Let $U = \{z \in \mathbb{C} / |z| \leq 1\}$ and, for $i = 1, \dots, n$, fix an embedding $e_i : U \longrightarrow N$ such that $e_i(0) = z_i$, $e_i(U) \cap e_j(U) = \emptyset$ if $i \neq j$ and $e_i(U) \cap \partial N = \emptyset$ for all i . If we remove the interior of each $e_i(U)$ (thus getting the surface $N_{g,b+n}$) and identify, for each $z \in \partial U$, $e_i(z)$ with $e_i(-z)$, we get a nonorientable surface of genus $g + n$ with b boundary components, that is to say a surface homeomorphic to $N_{g+n,b}$. Let us denote by $\gamma_i = e_i(S^1)$ the boundary of $e_i(U)$, and by μ_i its image in $N_{g+n,b}$: it is a one-sided simple closed curve.

Now, let h be an element of $\Gamma_{g,b}^n$. It can be represented by a homeomorphism $N_{g,b} \longrightarrow N_{g,b}$, still denoted h , such that

- (1) $h(e_i(z)) = e_i(z)$ if h preserves local orientation at z_i ;
- (2) $h(e_i(z)) = e_i(\bar{z})$ if h reverses local orientation at z_i .

Such a homeomorphism h commutes with the identification leading to $N_{g+n,b}$ and thus induces an element $\eta(h) \in \Gamma_{g+n,b}$. It is proved in [Sz2] that the map $\eta_{g,b}^n = \eta : \Gamma_{g,b}^n \longrightarrow \Gamma_{g+n,b}$ which sends h to $\eta(h)$ is well defined for $n = 1$, but the proof also works for $n > 1$. This homomorphism is called *blowup homomorphism*.

PROPOSITION 1 *The blowup homomorphism $\eta_{g,b}^n : \Gamma_{g,b}^n \longrightarrow \Gamma_{g+n,b}$ is injective if $(g + n, b) \neq (2, 0)$.*

REMARK 1 This result is proved in [Sz1] for $(g, b) = (0, 1)$, but the proof can be adapted in our case as follows.

Proof. Suppose that $h : N_{g,b} \rightarrow N_{g,b}$ is a homeomorphism satisfying $h(z_i) = z_i$ for all i and $\eta(h) : N_{g+n,b} \rightarrow N_{g+n,b}$ is isotopic to identity. Then h is isotopic to a map equal to identity on $e_i(U)$ for all i (otherwise, μ_i is isotopic to μ_i^{-1} since $\eta(h)(\mu_i)$ is isotopic to μ_i) and its restriction to $N_{g,b+n}$ is an element of the kernel of the natural map $\Gamma_{g,b+n} \rightarrow \Gamma_{g+n,b}$ induced by glueing a Möbius strip on n boundary components. However, this kernel is generated by the Dehn twists along the curves γ_i (see [St, theorem 3.6]). Now, any γ_i bounds a disc with one marked point in $N_{g,b}$: the corresponding Dehn twist is trivial in $\Gamma_{g,b}$ and therefore h is isotopic to identity. \square

2.3 Embedding $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$

Gluing a one-holed torus onto $b-1$ boundary components of $N_{g,b}$, we get $N_{g,b}$ as a subsurface of $N_{g+2(b-1),1}$. This inclusion induces a homomorphism $\chi_{g,b} : \Gamma_{g,b} \rightarrow \Gamma_{g+2(b-1),1}$ which is injective (see [St]). Thus, the composed map $\lambda_{g,b}^n = \chi_{g+n,b} \circ \eta_{g,b}^n : \Gamma_{g,b}^n \rightarrow \Gamma_{g+n+2(b-1),1}$ is also injective.

Recall that the mod p Torelli group $I_p(N_{g,1})$ is the subgroup of $\Gamma_{g,1}$ defined as the kernel of the action of $\Gamma_{g,1}$ on $H_1(N_{g,1}; \mathbb{Z}/p\mathbb{Z})$. In the following we will consider in particular the case of the mod 2 Torelli group $I_2(N_{g,1})$.

PROPOSITION 2 *If $b \geq 1$, $\lambda_{g,b}^n(P_n(N_{g,b}))$ is a subgroup of the Torelli subgroup $I_2(N_{g+n+2(b-1),1})$.*

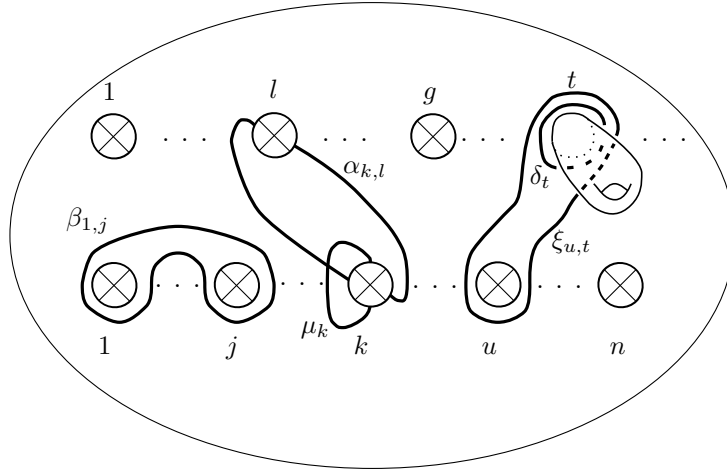


Figure 1: Image of the generators of $P_n(N_{g,b})$ in $\Gamma_{g+n+2(b-1),1}$

Proof. The image of the generators (see figures 2, 6 and theorem A) $(B_{i,j})_{1 \leq i < j \leq n}$, $(\rho_{k,l})_{\substack{1 \leq k \leq n \\ 1 \leq l \leq g}}$ and $(x_{u,t})_{\substack{1 \leq u \leq n \\ 1 \leq t \leq b-1}}$ of $P_n(N_{g,b})$ under $\lambda_{g,b}^n$ are respectively (see figure 1):

- * Dehn twist along curves $\beta_{i,j}$, which bound a subsurface homeomorphic to $N_{2,1}$;
- * crosscap slides $Y_{\mu_k, \alpha_{k,l}}$;
- * the product $\tau_{\xi_{u,t}} \tau_{\delta_t}^{-1}$ of Dehn twists along the bounding curves $\xi_{u,t}$ and δ_t .

According to [Sz2], all these elements are in the mod 2 Torelli subgroup $I_2(N_{g+n+2(b-1),1})$. \square

REMARK 2 The embedding provided in Proposition 2 does not hold for $I_p(N_{g+n+2(b-1),1})$, when $p \neq 2$: for example, the cross slide $Y_{\mu_k, \alpha_{k,l}}$ is not in the mod p Torelli subgroup since it sends μ_k to μ_k^{-1} .

2.4 Conclusion of the proof

We shall use the following result, which is a straightforward consequence of a similar result for mod p Torelli groups of orientable surfaces due to L. Paris [P]:

THEOREM 2 *Let $g \geq 1$. The mod p Torelli group $I_p(N_{g,1})$ is residually p -finite.*

Proof. We use Dehn-Nielsen-Baer Theorem (see for instance Theorem 5.15.3 of [CVZ]) which states that $\Gamma_{g,1}$ embeds in $\text{Aut}(\pi_1(N_{g,1}))$. Since $\pi_1(N_{g,1})$ is free we can apply Theorem 1.4 in [P] which claims that if G is a free group, its mod p Torelli group (i.e. the kernel of the canonical map from $\text{Aut}(G)$ to $GL(H_1(G, \mathbb{F}_p))$) is residually p -finite. Therefore $I_p(N_{g,1})$ is residually p -finite. \square

THEOREM 3 *Let $g \geq 1$, $b > 0$, $n \geq 1$. $P_n(N_{g,b})$ is residually 2-finite.*

Proof. The group $P_n(N_{g,b})$ is a subgroup of $I_2(N_{g+n+2(b-1),1})$ by Proposition 2 and by injectivity of the map $\lambda_{g,b}^n$. Then by Theorem 2 it follows that $P_n(N_{g,b})$ is residually 2-finite. \square

3 p -almost direct products

3.1 On residually p -finite groups

Let p be a prime number and G a group. If H is a subgroup of G , we denote by H^p the subgroup generated by $\{h^p / h \in H\}$. Following [P], we define the lower \mathbb{F}_p -linear central filtration $(\gamma_n^p G)_{n \in \mathbb{N}^*}$ of G by $\gamma_1^p G = G$ and, for $n \geq 1$, $\gamma_{n+1}^p G$ is the subgroup of G generated by $[G, \gamma_n^p G] \cup (\gamma_n^p G)^p$. Note that the subgroups $\gamma_n^p G$ are characteristic in G and that the quotient group $G/\gamma_2^p G$ is nothing but the first homology group $H_1(G; \mathbb{F}_p)$. The followings are proved in [P]:

- for $m, n \geq 1$, $[\gamma_m^p G, \gamma_n^p G] \subset \gamma_{m+n}^p G$;
- a finitely generated group G is a finite p -group if, and only if, there exists some $N \geq 1$ such that $\gamma_N^p G = \{1\}$;
- a finitely generated group G is residually p -finite if, and only if, $\bigcap_{n=1}^{+\infty} \gamma_n^p G = \{1\}$;

and clearly, if $f : G \longrightarrow G'$ is a group homomorphism, then $f(\gamma_n^p G) \subset \gamma_n^p G'$ for all $n \geq 1$.

DEFINITION 1 *Let $1 \longrightarrow A \xrightarrow{\quad} B \xrightarrow{\quad \lambda \quad} C \longrightarrow 1$ be a split exact sequence.*

- If the action of C induced on $H_1(A; \mathbb{Z})$ is trivial (i.e. the action is trivial on $A^{\text{Ab}} = A/[A, A]$), we say that B is an almost direct product of A and C .
- If the action of C induced on $H_1(A; \mathbb{F}_p)$ is trivial (i.e. the action is trivial on $A/\gamma_2^p A$), we say that B is a p -almost direct product of A and C .

Let us remark that, as in the case of almost direct products (Proposition 6.3 of [BGoGu]), the fact to be a p -almost direct product does not depend on the choice of the section.

PROPOSITION 3 *Let $1 \longrightarrow A \longrightarrow B \xrightarrow{\lambda} C \longrightarrow 1$ be a split exact sequence of groups. Let σ, σ' be sections for λ , and suppose that the induced action of C on A via σ on $H_1(A; \mathbb{F}_p)$ is trivial. Then the same is true for the section σ' .*

Proof. Let $a \in A$ and $c \in C$. By hypothesis, $\sigma(c) a (\sigma(c))^{-1} \equiv a \pmod{\gamma_2^p A}$. Let σ' be another section for λ . Then $\lambda \circ \sigma'(c) = \lambda \circ \sigma(c)$, and so $\sigma'(c) (\sigma(c))^{-1} \in \text{Ker}(\lambda)$. Thus there exists $a' \in A$ such that $\sigma'(c) = a' \sigma(c)$, and hence

$$\sigma'(c) a (\sigma'(c))^{-1} \equiv a' \sigma(c) a (\sigma(c))^{-1} a'^{-1} \equiv a' a a'^{-1} \equiv a \pmod{\gamma_2^p A}.$$

Thus the induced action of C on $H_1(A; \mathbb{F}_p)$ via σ' is also trivial. \square

The first goal of this section is to prove the following Theorem (see Theorem 3.1 in [FR] for an analogous result for almost direct products).

THEOREM 4 *Let $1 \longrightarrow A \longrightarrow B \xrightarrow{\lambda} C \longrightarrow 1$ be a split exact sequence where B is a p -almost direct product of A and C . Then, for all $n \geq 1$, one has a split exact sequence*

$$1 \longrightarrow \gamma_n^p A \longrightarrow \gamma_n^p B \xrightarrow{\lambda_n} \gamma_n^p C \longrightarrow 1$$

$\swarrow \sigma_n$

where λ_n and σ_n are restrictions of λ and σ .

We shall need the following preliminary result.

LEMMA 1 *Under the hypotheses of Theorem 4, one has, for all $m, n \geq 1$*

$$[\gamma_m^p C', \gamma_n^p A] \subset \gamma_{m+n}^p A$$

where C' denotes $\sigma(C)$.

Proof. First, we prove by induction on n that $[C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for all $n \geq 1$. The cas $n = 1$ corresponds to the hypotheses: the action of C on $H_1(A; \mathbb{F}_p) = A/\gamma_2^p A$ is trivial if, and only if, $[C', A] \subset \gamma_2^p A$. Thus, suppose that $[C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for some $n \geq 1$ and let us prove that $[C', \gamma_{n+1}^p A] \subset \gamma_{n+2}^p A$. In view of the definition of $\gamma_{n+1}^p A$, we have to prove that $[C', [A, \gamma_n^p A]] \subset \gamma_{n+2}^p A$ and $[C', (\gamma_n^p A)^p] \subset \gamma_{n+2}^p A$. For the first case, we use a classical result (see [MKS], theorem 5.2) which says

$$[C', [A, \gamma_n^p A]] = [\gamma_n^p A, [C', A]] [A, [\gamma_n^p A, C']].$$

We have just seen that $[C', A] \subset \gamma_2^p A$ thus $[\gamma_n^p A, [C', A]] \subset [\gamma_n^p A, \gamma_2^p A] \subset \gamma_{n+2}^p A$. Then, the induction hypotheses says that $[\gamma_n^p A, C'] \subset \gamma_{n+1}^p A$ thus $[A, [\gamma_n^p A, C']] \subset [A, \gamma_{n+1}^p A] \subset \gamma_{n+2}^p A$. The second case works as follows: for $c \in C'$ and $x \in \gamma_n^p A$, one has, using the fact that $[u, vw] = [u, w][u, v][u, v, w]$ (see [MKS])

$$[c, x^p] = [c, x][c, x^{p-1}][[c, x^{p-1}], x] = \cdots = [c, x]^p [[c, x], x] [[c, x^2], x] \cdots [[c, x^{p-1}], x].$$

Since $c \in C'$ and $x \in \gamma_n^p A$, one has $[c, x^i] \in [C', \gamma_n^p A] \subset \gamma_{n+1}^p A$ for all i , $1 \leq i \leq p-1$, which leads to $[c, x]^p \in (\gamma_{n+1}^p A)^p \subset \gamma_{n+2}^p A$ and $[[c, x^i], x] \in [\gamma_{n+1}^p A, A] \subset \gamma_{n+2}^p A$.

Now, we suppose that $[\gamma_m^p C', \gamma_n^p A] \subset \gamma_{m+n}^p A$ for some $m \geq 1$ and all $n \geq 1$ and prove that $[\gamma_{m+1}^p C', \gamma_n^p A] \subset \gamma_{m+n+1}^p A$. As above, there are two cases which work on the same way:

$$(i) \quad \begin{aligned} [[C', \gamma_m^p C'], \gamma_n^p A] &= [[\gamma_n^p A, C'], \gamma_m^p C'] [[\gamma_m^p C', \gamma_n^p A], C'] \\ &\subset [\gamma_{n+1}^p A, \gamma_m^p C'] [\gamma_{m+n}^p A, C'] \\ &\subset \gamma_{m+n+1}^p A. \end{aligned}$$

(ii) For $c \in \gamma_m^p C'$ and $x \in \gamma_n^p A$, one has

$$[c^p, x] = [c, [x, c^{p-1}]] [c^{p-1}, x] [c, x] = \cdots = [c, [x, c^{p-1}]] \cdots [c, [x, c]] [c, x]^p$$

which is an element of $\gamma_{m+n+1}^p A$ by induction hypotheses. \square

Proof of Theorem 4. Restrictions of λ and σ give rise to morphisms $\lambda_n : \gamma_n^p B \rightarrow \gamma_n^p C$ and $\sigma_n : \gamma_n^p C \rightarrow \gamma_n^p B$ satisfying $\lambda_n \circ \sigma_n = \text{Id}_{\gamma_n^p C}$, σ_n is onto and $\text{Ker}(\lambda_n) = A \cap \gamma_n^p B$. Thus, we need to prove that $A \cap \gamma_n^p B = \gamma_n^p A$ for all $n \geq 1$. Clearly one has $\gamma_n^p A \subset A \cap \gamma_n^p B$. In order to prove the converse inclusion, we follow the method developed in [FR] for almost semi-direct product and define $\tau : B \rightarrow B$ by $\tau(b) = (\sigma\lambda(b))^{-1}b$. This map has the following properties:

- (i) since $\lambda\sigma = \text{Id}_C$, $\tau(B) \subset A$;
- (ii) for $x \in B$, $\tau(x) = x$ if, and only if, $x \in A$;
- (iii) for $(b_1, b_2) \in B^2$, $\tau(b_1 b_2) = [\sigma\lambda(b_2), \tau(b_1)^{-1}] \tau(b_1) \tau(b_2)$;
- (iv) for $b \in B$, setting $a = \tau(b)$ and $c = \sigma\lambda(b)$, we get $b = ca$ with $c \in C' = \sigma(C)$ and $a \in A$, this decomposition being unique.

We claim that $\tau(\gamma_n^p B) \subset \gamma_n^p A$ for all $n \geq 1$. From this, we conclude easily the proof: if $x \in A \cap \gamma_n^p B$, then $x = \tau(x) \in \gamma_n^p A$.

One has $\tau(\gamma_1^p B) \subset \gamma_1^p A$. Suppose inductively that $\tau(\gamma_n^p B) \subset \gamma_n^p A$ for some $n \geq 1$ and let us prove that $\tau(\gamma_{n+1}^p B) \subset \gamma_{n+1}^p A$. Suppose first that x is an element of $\gamma_n^p B$. Then using (iii) we get:

$$\begin{aligned} \tau(x^p) &= [\sigma\lambda(x), \tau(x^{p-1})^{-1}] \tau(x^{p-1}) \tau(x) \\ &\vdots \\ &= [\sigma\lambda(x), \tau(x^{p-1})^{-1}] [\sigma\lambda(x), \tau(x^{p-2})^{-1}] \cdots [\sigma\lambda(x), \tau(x)^{-1}] \tau(x)^p. \end{aligned}$$

Since $\sigma\lambda(x) \in \gamma_n^p C'$ and, by induction hypotheses, $\tau(x^i) \in \gamma_n^p A$ for $1 \leq i \leq p-1$, we get $\tau(x^p) \in [\gamma_n^p C', \gamma_n^p A] \cdot (\gamma_n^p A)^p \subset \gamma_{n+1}^p A$ by lemma 1 : this prove that $\tau((\gamma_n^p B)^p) \subset \gamma_{n+1}^p A$. Next, let $b \in B$ and $x \in \gamma_n^p B$. Setting $a = \tau(b) \in A$, $y = \tau(x) \in \gamma_n^p A$ by induction hypotheses, $c = \sigma\lambda(b) \in C'$ and $z = \sigma\lambda(x) \in \gamma_n^p C'$, we get

$$\begin{aligned} \tau([b, x]) &= \left(\sigma\lambda([b, x]) \right)^{-1} [b, x] \\ &= [\sigma\lambda(b), \sigma\lambda(x)]^{-1} [b, x] \\ &= [c, z]^{-1} [ca, zy] = [z, c] a^{-1} c^{-1} y^{-1} z^{-1} c a z y \\ &= [z, c] (a^{-1} c^{-1} y^{-1} c y a) (a^{-1} y^{-1} c^{-1} z^{-1} c z y a) (a^{-1} y^{-1} z^{-1} a z y) \\ &= [z, c] (a^{-1} [c, y] a) (a^{-1} y^{-1} [c, z] y a) (a^{-1} y^{-1} a y) (y^{-1} a^{-1} z^{-1} a z y) \\ &= [z, c] (a^{-1} [c, y] a) (a^{-1} y^{-1} [c, z] y a) [a, y] (y^{-1} [a, z] y) \\ &= \left[[c, z], (a^{-1} [y, c] a) \right] (a^{-1} [c, y] a) [z, c] (a^{-1} y^{-1} [c, z] y a) [a, y] (y^{-1} [a, z] y) \end{aligned}$$

$$= \left[[c, z], (a^{-1}[y, c]a) \right] (a^{-1}[c, y]a) \left[[c, z], ya \right] [a, y] (y^{-1}[a, z]y).$$

Now, $[c, z] \in [C', \gamma_n^p C'] \subset \gamma_{n+1}^p C'$, $[y, c] \in [\gamma_n^p A, C'] \subset \gamma_{n+1}^p A$ (lemma 1) thus $[c, z], (a^{-1}[y, c]a) \in \gamma_{n+1}^p A$. Then, $[c, z], ya \in [\gamma_{n+1}^p C', A] \subset \gamma_{n+1}^p A$, $[a, y] \in [A, \gamma_n^p A] \subset \gamma_{n+1}^p A$ and $[a, z] \in [A, \gamma_n^p C'] \subset \gamma_{n+1}^p A$. Thus, $\tau([b, x]) \in \gamma_{n+1}^p A$ and $\tau([B, \gamma_n^p B]) \subset \gamma_{n+1}^p A$. \square

COROLLARY 2 *Let $1 \longrightarrow A \xrightarrow{\quad} B \xrightarrow{\quad \lambda \quad} C \longrightarrow 1$ be a split exact sequence such that B is a p -almost direct product of A and C . If A and C are residually p -finite, then B is residually p -finite.*

3.2 Augmentation ideals

Given a group G and $\mathbb{K} = \mathbb{Z}$ or \mathbb{F}_2 we will denote by $\mathbb{K}[G]$ the group ring of G over \mathbb{K} and by $\overline{\mathbb{K}[G]}$ the augmentation ideal of G . The group ring $\mathbb{K}[G]$ is filtered by the powers $\overline{\mathbb{K}[G]}^j$ of $\overline{\mathbb{K}[G]}$ and we can define the associated graded algebra $gr(\mathbb{K}[G]) = \bigoplus \overline{\mathbb{K}[G]}^j / \overline{\mathbb{K}[G]}^{j+1}$.

The following theorem provides a decomposition formula for the augmentation ideal of a 2-almost direct product (see Theorem 3.1 in [Pa] for an analogous in the case of almost direct products).

Let $A \rtimes C$ be a semi-direct product between two groups A and C . It is a classical result that the map $a \otimes c \mapsto ac$ induces a \mathbb{K} -isomorphism from $\mathbb{K}[A] \otimes \mathbb{K}[C]$ to $\mathbb{K}[A \rtimes C]$. Identifying these two \mathbb{K} -modules, we have the following:

THEOREM 5 *If $A \rtimes C$ is a 2-almost direct product then :*

$$\overline{\mathbb{F}_2[A \rtimes C]}^k = \sum_{i+h=k} \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h \quad \text{for all } k.$$

Proof. We sketch the proof which is almost verbatim the same as the proof of Theorem 3.1 in [Pa]. Let $R_k = \sum_{i+h=k} \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h$; R_k is a descending filtration on $\mathbb{F}_2[A] \otimes \mathbb{F}_2[C]$, and with the above identification,

we get that $R_k \subset \overline{\mathbb{F}_2[A \rtimes C]}^k$. To verify the other inclusion we have to check that $\prod_{j=1}^k (a_j c_j - 1) \in R_k$

for every a_1, \dots, a_k in A and c_1, \dots, c_k in C . Actually it is enough to verify that $e = \prod_{j=1}^k (e_j - 1) \in R_k$

either $e_j \in A$ or $e_j \in C$ (see Theorem 3.1 in [Pa] for a proof of this fact): we call e a *special* element. We associate to a special element e an element in $\{0, 1\}^k$: let $type(e) = (\delta(e_1), \dots, \delta(e_k))$ where $\delta(e_j) = 0$ if $e_j \in A$ and $\delta(e_j) = 1$ if $e_j \in C$. We will say that the special element e is *standard* if

$$type(e) = (\overbrace{0, \dots, 0}^i, \overbrace{1, \dots, 1}^h)$$

In this case $e \in \overline{\mathbb{F}_2[A]}^i \otimes \overline{\mathbb{F}_2[C]}^h \subset R_k$ and we are done. We claim that we can reduce all special elements to linear combinations of standard elements. If e is not standard, then it must be of the form

$$e = \prod_{i=1}^r (a_i - 1) \prod_{j=1}^s (c_j - 1)(c - 1)(a - 1) \prod_{l=1}^t (e_l - 1)$$

where $a_1, \dots, a_r, a \in A$, $c_1, \dots, c_s, c \in A$, $\tilde{e} = \prod_{l=1}^t (e_l - 1)$ is special and $r + s + t + 2 = k$. Therefore

$$\text{type}(e) = (\overbrace{0, \dots, 0}^r, \overbrace{1, \dots, 1}^s, 1, 0, \delta(e_1), \dots, \delta(e_t)).$$

Now we can use the assumption that $A \rtimes C$ is a 2-almost direct product to claim that one has commutation relations in $\mathbb{Z}[A \rtimes C]$ expressing the difference $(c - 1)(a - 1) - (a - 1)(c - 1)$ as a linear combination of terms of the form

$$(a' - 1)(a'' - 1)c \quad \text{with } a', a'' \in A$$

for any $a \in A$ and $c \in C$. In fact,

$$(c - 1)(a - 1) - (a - 1)(c - 1) = ca - ac = (cac^{-1}a^{-1} - 1)ac = (f - 1)ac$$

where $f = [c^{-1}, a^{-1}] \in [C, A] \subset \gamma_2^2(A)$ by lemma 1. We can decompose f as $f = h_1 k_1 \cdots h_m k_m$ where, for $j = 1, \dots, m$, h_j belongs to $[A, A]$ and $k_j = (k'_j)^2$ for some $k'_j \in A$. One knows (see for instance [Ch] p. 194) that for $j = 1, \dots, m$ $(h_j - 1)$ is a linear combination of terms of the form

$$(h'_j - 1)(h''_j - 1)\alpha_j \quad \text{with } h'_j, h''_j, \alpha_j \in A.$$

On the other hand for $j = 1, \dots, m$ we have also that

$$(k_j - 1) = (k'_j - 1)(k'_j - 1) \quad \text{with } k'_j \in A \quad \text{since the coefficients are } \mathbb{F}_2.$$

Then, recalling that $(hk - 1) = (h - 1)k + (k - 1)$ for any $h, k \in A$, we can conclude that $f - 1$ can be rewritten as a linear combination of terms of the form

$$(f' - 1)(f'' - 1)\alpha \quad \text{with } f', f'', \alpha \in A$$

and that $(c - 1)(a - 1) - (a - 1)(c - 1)$ is a linear combination of terms of the form

$$(f' - 1)(f'' - 1)\alpha c \quad \text{with } f', f'', \alpha \in A.$$

Rewriting $(f'' - 1)\alpha$ as $(f''\alpha - 1) - (\alpha - 1)$ we obtain that the difference $(c - 1)(a - 1) - (a - 1)(c - 1)$ can be seen as a linear combination of terms of the form

$$(a' - 1)(a'' - 1)c \quad \text{with } a', a'' \in A.$$

Therefore e can be rewritten as a sum whose first term is the special element

$$e' = \prod_{i=1}^r (a_i - 1) \prod_{j=1}^s (c_j - 1)(a - 1)(c - 1) \prod_{l=1}^t (e_l - 1)$$

and whose second term is a linear combination of elements of the form $e''c$ where

$$e'' = \prod_{i=1}^r (a_i - 1) \prod_{j=1}^s (c_j - 1)(a' - 1)(a'' - 1) \prod_{l=1}^t (ce_l c^{-1} - 1)c.$$

Using the lexicographic order from the left, one has $\text{type}(e) > \text{type}(e')$ and $\text{type}(e) > \text{type}(e'')$.

By induction on the lexicographic order we infer that e' and e'' belong to R_k : since $R_k \cdot c \subset R_k$ for any $c \in C$ it follows that e belongs to R_k and we are done. \square

4 The closed case

4.1 A presentation of $P_n(N_g)$ and induced identities

We recall a group presentation of $P_n(N_g)$ given in [GG3]: the geometric interpretation of generators is provided in Figure 2.

THEOREM 6 ([GG3]) *For $g \geq 2$ and $n \geq 1$, $P_n(N_g)$ has the following presentation:*

generators: $(B_{i,j})_{1 \leq i < j \leq n}$ and $(\rho_{k,l})_{\substack{1 \leq k \leq n \\ 1 \leq l \leq g}}$.

relations: (a) for all $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$,

$$B_{r,s} B_{i,j} B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j & (a_1) \\ B_{i,j}^{-1} B_{r,j}^{-1} B_{i,j} B_{r,j} B_{i,j} & \text{if } r < i = s < j & (a_2) \\ B_{s,j}^{-1} B_{i,j} B_{s,j} & \text{if } i = r < s < j & (a_3) \\ B_{s,j}^{-1} B_{r,j}^{-1} B_{s,j} B_{r,j} B_{i,j} B_{r,j}^{-1} B_{s,j}^{-1} B_{r,j} B_{s,j} & \text{if } r < i < s < j & (a_4) \end{cases}$$

(b) for all $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$,

$$\rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k < l & (b_1) \\ \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k}^2 & \text{if } k = l & (b_2) \\ \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{i,j}^{-1} \rho_{j,l} B_{i,j} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k} & \text{if } k > l & (b_3) \end{cases}$$

(c) for all $1 \leq i \leq n$, $\rho_{i,1}^2 \cdots \rho_{i,g}^2 = T_i$ where $T_i = B_{1,i} \cdots B_{i-1,i} B_{i,i+1} \cdots B_{i,n}$ (c)

(d) for all $1 \leq i < j \leq n$, $1 \leq k \leq n$, $k \neq j$ and $1 \leq l \leq g$,

$$\rho_{k,l} B_{i,j} \rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k < i \text{ or } j < k & (d_1) \\ \rho_{j,l}^{-1} B_{i,j}^{-1} \rho_{j,l} & \text{if } k = i & (d_2) \\ \rho_{j,l}^{-1} B_{k,j}^{-1} \rho_{j,l} B_{k,j}^{-1} B_{i,j} B_{k,j} \rho_{j,l}^{-1} B_{k,j} \rho_{j,l} & \text{if } i < k < j & (d_3) \end{cases}$$

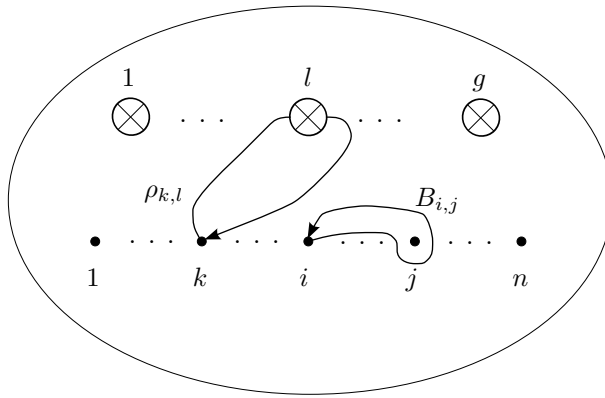


Figure 2: Generators of $P_n(N_g)$

For $1 \leq k \leq g$, let us consider the element a_k in $P_n(N_g)$ given by $a_k = \rho_{k,g-1} \rho_{k,g}$ and set $U = a_n \cdots a_2$.

LEMMA 2 *The following relations holds in $P_n(N_g)$:*

- (1) $[\rho_{i,k}, \rho_{j,k}^{-1}] = B_{i,j}^{-1}$ for $1 \leq i < j \leq n$ and $1 \leq k \leq g$; (e)
- (2) U commutes with $\rho_{1,l}$ for $1 \leq l \leq g-2$; (f₁)
- (3) $[\rho_{1,g-1}, U^{-1}] = T_1^{-1}$; (f₂)
- (4) $a_k a_j a_k^{-1}$ commutes with $B_{i,k}$ for $1 \leq i < j < k \leq n$; (g)
- (5) $a_n a_{n-1} \cdots a_1$ commutes with $B_{j,k}$ for $1 \leq j < k \leq n$; (h)
- (6) U commutes with $B_{i,j}$ for $2 \leq i < j \leq n$; (i)
- (7) $a_n a_{n-1} \cdots a_1$ commutes with T_1 ; (j)
- (8) T_1 commutes with $B_{j,k}$ for $2 \leq j < k \leq n$; (k)

Proof. Some of these identities can easily be verified drawing corresponding braids. This is the case for example for the first, the fourth and the eighth ones (see figure 3, 4 and 5). Let us give an algebraic proof for the others.

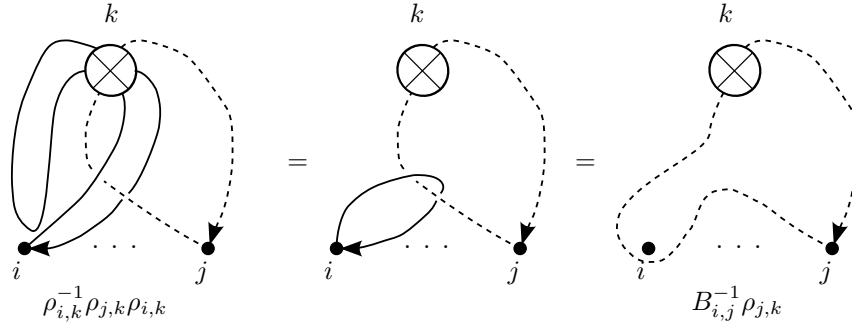


Figure 3: identity (e)

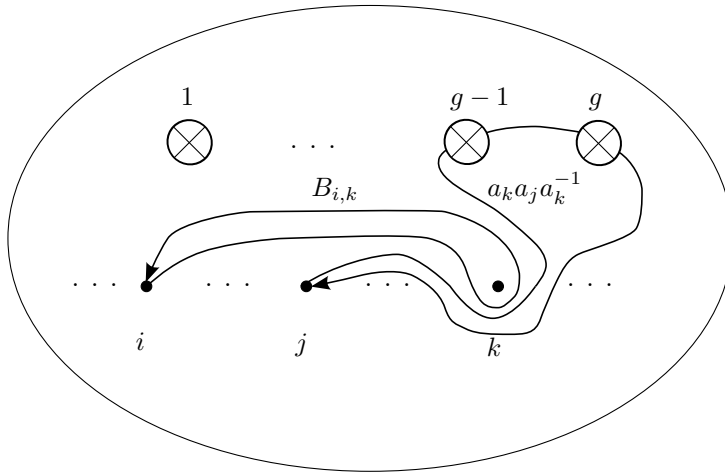


Figure 4: identity (g)

- (2) This is a direct consequence of the definitions of the a_i 's and U , and relations (b₁).

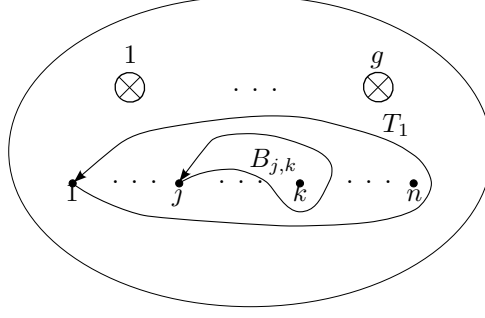


Figure 5: identity (k)

(3) By relation (b₁), $\rho_{1,g-1}$ commutes with $\rho_{j,g}$ for $2 \leq j \leq n$. Thus, using relation (e), we get:

$$\begin{aligned}
 \rho_{1,g-1}^{-1} U \rho_{1,g-1} &= \rho_{1,g-1}^{-1} a_n \cdots a_2 \rho_{1,g-1} \\
 &= \rho_{1,g-1}^{-1} (\rho_{n,g-1} \rho_{n,g}) \cdots (\rho_{2,g-1} \rho_{2,g}) \rho_{1,g-1} \\
 &= (B_{1,n}^{-1} \rho_{n,g-1} \rho_{n,g}) \cdots (B_{1,2}^{-1} \rho_{2,g-1} \rho_{2,g}) \\
 &= B_{1,n}^{-1} B_{1,n-1}^{-1} \cdots B_{1,2}^{-1} (\rho_{n,g-1} \rho_{n,g}) \cdots (\rho_{2,g-1} \rho_{2,g}) \quad \text{by (d}_1\text{)} \\
 &= T_1^{-1} U.
 \end{aligned}$$

(5) Let j and k be integers such that $1 \leq j < k \leq n$. By (d₁), a_1, \dots, a_{j-1} commute with $B_{j,k}$. Then, one has

$$\begin{aligned}
 a_j B_{j,k} a_j^{-1} &= \rho_{j,g-1} \rho_{j,g} B_{j,k} \rho_{j,g}^{-1} \rho_{j,g-1}^{-1} \\
 &= \rho_{j,g-1} \rho_{k,g}^{-1} B_{j,k}^{-1} \rho_{k,g} \rho_{j,g-1}^{-1} \quad \text{by (d}_2\text{)} \\
 &= \rho_{k,g}^{-1} \rho_{j,g-1} B_{j,k}^{-1} \rho_{j,g-1}^{-1} \rho_{k,g} \quad \text{by (b}_1\text{)} \\
 &= \rho_{k,g}^{-1} \rho_{k,g-1}^{-1} B_{j,k} \rho_{k,g-1} \rho_{k,g} \quad \text{by (d}_2\text{)} \\
 &= a_k^{-1} B_{j,k} a_k
 \end{aligned}$$

and we get

$$\begin{aligned}
 a_n \cdots a_1 B_{j,k} a_1^{-1} \cdots a_n^{-1} &= a_n \cdots a_{k+1} a_k a_{k-1} \cdots a_{j+1} a_k^{-1} B_{j,k} a_k a_{j+1}^{-1} \cdots a_{k-1}^{-1} a_k^{-1} a_{k+1}^{-1} \cdots a_n^{-1} \\
 &= a_n \cdots a_{k+1} B_{j,k} a_{k+1}^{-1} \cdots a_n^{-1} \quad \text{by (g)} \\
 &= B_{j,k} \quad \text{by (d}_1\text{)}.
 \end{aligned}$$

(6) By (d₁), $a_1 = \rho_{1,g-1} \rho_{1,g}$ commutes with $B_{i,j}$ for $2 \leq i < j \leq n$. Thus, relation (i) is a direct consequence of (h).

(7) A direct consequence of (h) since $T_1 = B_{1,2} \cdots B_{1,n}$.

□

4.2 The pure braid group $P_n(N_g)$ is residually 2-finite

Following [GG1], one has, for $g \geq 2$, a split exact sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \xrightarrow{\mu} P_n(N_g) \xrightarrow{\lambda} P_1(N_g) = \pi_1(N_g) \longrightarrow 1 \quad (1)$$

where λ is induced by the map which forgets all strands except the first one, and μ is defined by capping the boundary component by a disc with one marked point (the first strand in $P_n(N_g)$). According to the definition of μ and to Theorem A, $\text{Im}(\mu)$ is generated by $\{\rho_{i,k}, 2 \leq i \leq n, 1 \leq k \leq g\} \cup \{B_{i,j}, 2 \leq i < j \leq n\}$.

The section given in [GG1] is geometric, *i.e.* it is induced by a crossed section at the level of fibrations. In order to study the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$, we need an algebraic one. Recall that $\pi_1(N_g)$ has a group presentation with generators p_1, \dots, p_g and the single relation $p_1^2 \cdots p_g^2 = 1$. We define the set map $\sigma : \pi_1(N_g) \rightarrow P_n(N_g)$ by setting

$$\sigma(p_i) = \begin{cases} \rho_{1,i} & \text{for } 1 \leq i \leq g-3, \\ \rho_{1,g-2}U^{-1} & \text{for } i = g-2, \\ U\rho_{1,g-1} & \text{for } i = g-1, \\ \rho_{1,g}T_1^{-1} & \text{for } i = g. \end{cases}$$

PROPOSITION 4 *The map σ is a well defined homomorphism satisfying $\lambda \circ \sigma = \text{Id}_{\pi_1(N_g)}$.*

Proof. Since $\lambda(\rho_{1,i}) = p_i$ for all $1 \leq i \leq g$ and $\lambda(U) = \lambda(T_1) = 1$, one has clearly $\lambda\sigma = \text{Id}_{\pi_1(N_g)}$ if σ is a group homomorphism. Thus, we have just to prove that $\sigma(p_1)^2 \cdots \sigma(p_g)^2 = 1$:

$$\begin{aligned} \sigma(p_1)^2 \cdots \sigma(p_g)^2 &= (\rho_{1,1}^2 \cdots \rho_{1,g-3}^2)(\rho_{1,g-2}U^{-1})^2(U\rho_{1,g-1})^2(\rho_{1,g}T_1^{-1})^2 \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \underbrace{U^{-1}\rho_{1,g-2}\rho_{1,g-1}Ua_1T_1^{-1}\rho_{1,g}T_1^{-1}}_{\text{by (f}_1\text{)}} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \underbrace{U^{-1}\rho_{1,g-1}Ua_1T_1^{-1}\rho_{1,g}T_1^{-1}}_{\text{by (f}_2\text{)}} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1}^2 U^{-1}T_1 \underbrace{Ua_1T_1^{-1}\rho_{1,g}T_1^{-1}}_{\text{by (j)}} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1}^2 U^{-1}T_1 T_1^{-1} Ua_1 \rho_{1,g} T_1^{-1} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1}^2 a_1 \rho_{1,g} T_1^{-1} \\ &= \rho_{1,1}^2 \cdots \rho_{1,g-3}^2 \rho_{1,g-2}^2 \rho_{1,g-1}^2 \rho_{1,g}^2 T_1^{-1} \\ &= 1 \quad \text{by (c).} \end{aligned}$$

□

So, the exact sequence (1) splits. In order to apply Theorem 4, we have to prove that the action of $\pi_1(N_g)$ on $P_{n-1}(N_{g,1})$ is trivial on $H_1(P_{n-1}(N_{g,1}); \mathbb{F}_2)$. This is the claim of the following proposition.

PROPOSITION 5 *For all $x \in \text{Im}(\sigma)$ and $a \in \text{Im}(\mu)$, one has $[x^{-1}, a^{-1}] = xax^{-1}a^{-1} \in \gamma_2^2(\text{Im}(\mu))$.*

Proof. It is enough to prove the result for $a \in \{B_{j,k}, 2 \leq j < k \leq n\} \cup \{\rho_{j,l}, 2 \leq j \leq n \text{ and } 1 \leq l \leq g\}$ and $x \in \{\sigma(p_1), \dots, \sigma(p_g)\}$, respectively sets of generators of $\text{Im}(\mu)$ and $\text{Im}(\sigma)$. Suppose first that $2 \leq j < k \leq n$. One has:

- $[\sigma(p_i)^{-1}, B_{j,k}^{-1}] = [\rho_{1,i}^{-1}, B_{j,k}^{-1}] = 1$ for $1 \leq i \leq g-3$ by (d₁);
- $[\sigma(p_{g-2})^{-1}, B_{j,k}^{-1}] = [U\rho_{1,g-2}^{-1}, B_{j,k}^{-1}] = 1$ by (d₁) and (i);
- $[\sigma(p_{g-1})^{-1}, B_{j,k}^{-1}] = [\rho_{1,g-1}^{-1}U^{-1}, B_{j,k}^{-1}] = 1$ by (d₁) and (i);
- $[\sigma(p_g)^{-1}, B_{j,k}^{-1}] = [T_1\rho_{1,g}^{-1}, B_{j,k}^{-1}] = 1$ by (d₁) and (k).

Now, let j and l be integers such that $2 \leq j \leq n$ and $1 \leq l \leq g$ and let us first prove that $[\rho_{1,i}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$ for all $i, 1 \leq i \leq n$:

- this is clear for $i < l$ by (b₁);
- for $i = l$, the relation (b₂) gives $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] = \rho_{j,l}^{-1} B_{1,j}^{-1} \rho_{j,l}$. But

$$B_{1,j}^{-1} = B_{2,j} \cdots B_{j-1,j} B_{j,j+1} \cdots B_{j,n} \rho_{j,g}^{-2} \cdots \rho_{j,1}^{-2} \quad (\text{relation (c)})$$

is an element of $\gamma_2^2(\text{Im}(\mu))$ by (e), thus we get $[\rho_{1,l}^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$.

- If $l < i$ then $[\rho_{1,i}^{-1}, \rho_{j,i}^{-1}] = [B_{1,j} \rho_{j,i}^{-1} B_{1,j}^{-1} \rho_{j,i}, \rho_{j,i}^{-1}]$ by (b₃) so $[\rho_{1,i}^{-1}, \rho_{j,i}^{-1}] \in \gamma_2(\text{Im}(\mu))$ since $\rho_{j,i}$, $\rho_{j,i}^{-1}$ and $B_{1,j}$ are elements of $\text{Im}(\mu)$.

From this, we deduce the following facts.

$$(1) [\sigma(p_i)^{-1}, \rho_{j,i}^{-1}] \in \gamma_2^2(\text{Im}(\mu)) \text{ for } i \leq g-3 \text{ since } \sigma(p_i) = \rho_{1,i}.$$

(2) $[\sigma(p_{g-2})^{-1}, \rho_{j,l}^{-1}] = [U \rho_{1,g-2}^{-1}, \rho_{j,l}^{-1}] = \rho_{1,g-2} [U, \rho_{j,l}^{-1}] \rho_{1,g-2}^{-1} [\rho_{1,g-2}^{-1}, \rho_{j,l}^{-1}]$. But U and $\rho_{j,l}^{-1}$ are elements of $\text{Im}(\mu)$ thus $[U, \rho_{j,l}^{-1}] \in \gamma_2(\text{Im}(\mu)) \subset \gamma_2^2(\text{Im}(\mu))$. Consequently, $\rho_{1,g-2} [U, \rho_{j,l}^{-1}] \rho_{1,g-2}^{-1} \in \gamma_2^2(\text{Im}(\mu))$ since $\gamma_2^2(\text{Im}(\mu))$ is a characteristic subgroup of $\text{Im}(\mu)$ and $\text{Im}(\mu)$ is normal in $P_n(N_g)$. Thus, we get $[\sigma(p_{g-2})^{-1}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$. In the same way, one has

$$[\sigma(p_{g-1})^{-1}, \rho_{j,l}^{-1}] = [\rho_{1,g-1}^{-1} U^{-1}, \rho_{j,l}^{-1}] = U [\rho_{1,g-1}^{-1}, \rho_{j,l}^{-1}] U^{-1} [U, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu)).$$

(4) At last,

$$[\sigma(p_g)^{-1}, \rho_{j,l}^{-1}] = [T_1 \rho_{1,g}^{-1}, \rho_{j,l}^{-1}] = \rho_{1,g} [T_1, \rho_{j,l}^{-1}] \rho_{1,g}^{-1} [\rho_{1,g}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$$

since $T_1, \rho_{j,l} \in \text{Im}(\mu)$ and $[\rho_{1,g}, \rho_{j,l}^{-1}] \in \gamma_2^2(\text{Im}(\mu))$. □

We are now ready to prove the main result of this section

THEOREM 7 *For all $g \geq 2$ and $n \geq 1$, the pure braid group $P_n(N_g)$ is residually 2-finite.*

Proof. Proposition 4 says that the sequence

$$1 \longrightarrow P_{n-1}(N_{g,1}) \longrightarrow P_n(N_g) \twoheadrightarrow \pi_1(N_g) \longrightarrow 1$$

splits. Now $P_{n-1}(N_{g,1})$ is residually 2-finite (Theorem 3). It is proved in [B1] and [B2] that $\pi_1(N_g)$ is residually free for $g \geq 4$, so it is residually 2-finite. This result is proved in [LM] (lemma 8.9) for $g = 3$. When $g = 2$, $\pi_1(N_2)$ has presentation $\langle a, b \mid aba^{-1} = b^{-1} \rangle$ so is a 2-almost direct product of \mathbb{Z} by \mathbb{Z} . Since \mathbb{Z} is residually 2-finite, $\pi_1(N_2)$ is residually 2-finite by corollary 2. So, using Proposition 5 and Corollary 2, we can conclude that $P_n(N_g)$ is residually 2-finite. □

REMARK 3 It follows from the proof of Theorem 7 that, when $g > 2$, if $P_n(N_{g,1})$ is residually p -finite for some $p \neq 2$ then the pure braid group $P_n(N_g)$ is also residually p -finite.

4.3 The case $P_n(\mathbb{RP}^2)$

The main reason to exclude $N_1 = \mathbb{RP}^2$ in Theorem 7 is that the exact sequence (1) doesn't exist in this case, but forgetting at most $n - 2$ strands we get the following exact sequence ($1 \leq m \leq n - 2$; see [VB])

$$1 \longrightarrow P_m(N_{1,n-m}) \longrightarrow P_n(\mathbb{RP}^2) \longrightarrow P_{n-m}(N_g) \longrightarrow 1 .$$

This sequence splits if, and only if $n = 3$ and $m = 1$ (see [GG2]). Thus, what we know is the following:

- $P_1(\mathbb{RP}^2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$: $P_1(\mathbb{RP}^2)$ is a 2-group.
- $P_2(\mathbb{RP}^2) = Q_8$, the quaternion group (see [VB]): $P_2(\mathbb{RP}^2)$ is a 2-group.
- One has the exact sequence

$$1 \longrightarrow P_1(N_{1,2}) \longrightarrow P_3(\mathbb{RP}^2) \longrightarrow P_2(\mathbb{RP}^2) \longrightarrow 1$$

where $P_1(N_{1,2}) = \pi_1(N_{1,2})$ is a free group of rank 2, thus is residually 2-finite. Since $P_2(\mathbb{RP}^2)$ is 2-finite, we can conclude that $P_3(\mathbb{RP}^2)$ is residually 2-finite using lemma 1.5 of [Gr].

Appendix: a group presentation for $P_n(N_{g,b})$

Here we apply classical method (see [B, GG3]) to give a presentation of the n^{th} pure braid group of a nonorientable surface with boundary. Since $b \geq 1$, we'll see $N_{g,b}$ as a disc D^2 with $g + b - 1$ open discs removed and g Möbius strips glued on g boundary components so obtained (see figure 6).

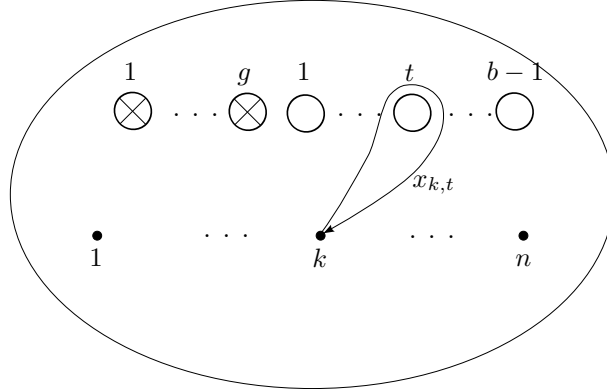


Figure 6: Generators $x_{k,t}$ for $P_n(N_{g,b})$, $b \geq 1$

THEOREM A For $g \geq 1$, $b \geq 1$ and $n \geq 1$, $P_n(N_{g,b})$ has the following presentation:

generators: $(B_{i,j})_{1 \leq i < j \leq n}$, $(\rho_{k,t})_{\substack{1 \leq k \leq n \\ 1 \leq t \leq g}}$ and $(x_{u,t})_{\substack{1 \leq u \leq n \\ 1 \leq t \leq b-1}}$.

relations : (a) for all $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$,

$$B_{r,s} B_{i,j} B_{r,s}^{-1} = \begin{cases} B_{i,j} & \text{if } i < r < s < j \text{ or } r < s < i < j & (a_1) \\ B_{i,j}^{-1} B_{r,j}^{-1} B_{i,j} B_{r,j} B_{i,j} & \text{if } r < i = s < j & (a_2) \\ B_{s,j}^{-1} B_{i,j} B_{s,j} & \text{if } i = r < s < j & (a_3) \\ B_{s,j}^{-1} B_{r,j}^{-1} B_{s,j} B_{r,j} B_{i,j} B_{r,j}^{-1} B_{s,j}^{-1} B_{r,j} B_{s,j} & \text{if } r < i < s < j & (a_4) \end{cases}$$

(b) for all $1 \leq i < j \leq n$ and $1 \leq k, l \leq g$,

$$\rho_{i,k} \rho_{j,l} \rho_{i,k}^{-1} = \begin{cases} \rho_{j,l} & \text{if } k < l & (b_1) \\ \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k}^2 & \text{if } k = l & (b_2) \\ \rho_{j,k}^{-1} B_{i,j}^{-1} \rho_{j,k} B_{i,j}^{-1} \rho_{j,l} B_{i,j} \rho_{j,k}^{-1} B_{i,j} \rho_{j,k} & \text{if } k > l & (b_3) \end{cases}$$

(d) for all $1 \leq i < j \leq n$, $1 \leq k \leq n$, $k \neq j$ and $1 \leq l \leq g$,

$$\rho_{k,l} B_{i,j} \rho_{k,l}^{-1} = \begin{cases} B_{i,j} & \text{if } k < i \text{ or } j < k & (d_1) \\ \rho_{j,l}^{-1} B_{i,j}^{-1} \rho_{j,l} & \text{if } k = i & (d_2) \\ \rho_{j,l}^{-1} B_{k,j}^{-1} \rho_{j,l} B_{k,j}^{-1} B_{i,j} B_{k,j} \rho_{j,l}^{-1} B_{k,j} \rho_{j,l} & \text{if } i < k < j & (d_3) \end{cases}$$

(l) for all $1 \leq i < j \leq n$, $1 \leq u \leq n$, $u \neq j$ and $1 \leq t \leq b-1$,

$$x_{u,t} B_{i,j} x_{u,t}^{-1} = \begin{cases} B_{i,j} & \text{if } u < i \text{ or } j < u & (l_1) \\ x_{j,t}^{-1} B_{i,j} x_{j,t} & \text{if } u = i & (l_2) \\ x_{j,t}^{-1} B_{u,j} x_{j,t} B_{u,j}^{-1} B_{i,j} B_{u,j} x_{j,t}^{-1} B_{u,j}^{-1} x_{j,t} & \text{if } i < u < j & (l_3) \end{cases}$$

(m) for all $1 \leq k, u \leq n$, $k \neq u$, $1 \leq l \leq g$ and $1 \leq t \leq b-1$,

$$x_{u,t} \rho_{k,l} x_{u,t}^{-1} = \begin{cases} \rho_{k,l} & \text{if } k < u & (m_1) \\ x_{k,t}^{-1} B_{u,k} x_{k,t} B_{u,j}^{-1} \rho_{k,l} B_{u,k} x_{k,t}^{-1} B_{u,k}^{-1} x_{k,t} & \text{if } u < k & (m_2) \end{cases}$$

(n) for all $1 \leq i < j \leq n$ and $1 \leq s, t \leq b-1$,

$$x_{i,t} x_{j,s} x_{i,t}^{-1} = \begin{cases} x_{j,s} & \text{if } t < s & (n_1) \\ x_{j,t}^{-1} B_{i,j} x_{j,t} B_{i,j}^{-1} x_{j,t} & \text{if } t = s & (n_2) \\ x_{j,t}^{-1} B_{i,j} x_{j,t} B_{i,j}^{-1} x_{j,s} B_{i,j} x_{j,t}^{-1} B_{i,j}^{-1} x_{j,t} & \text{if } s < t & (n_3) \end{cases}$$

Proof. The proof works by induction and generalizes those of [GG3] (closed non orientable case) and [B] (orientable case, possibly with boundary components). It uses the following short exact sequence obtained by forgetting the last strand (see [FN]):

$$1 \longrightarrow \pi_1(N_{g,b} \setminus \{z_1, \dots, z_n\}, z_{n+1}) \xrightarrow{\alpha} P_{n+1}(N_{g,b}) \xrightarrow{\beta} P_n(N_{g,b}) \longrightarrow 1.$$

The presentation is correct for $n = 1$: $P_1(N_{g,b}) = \pi_1(N_{g,b})$ is free on the $\rho_{1,l}$'s and $x_{1,t}$'s for $1 \leq l \leq g$ and $1 \leq t \leq b-1$. Suppose inductively that $P_n(N_{g,b})$ has the given presentation. Then, observe that $\{B_{i,n+1} / 1 \leq i \leq n\} \cup \{\rho_{n+1,l} / 1 \leq l \leq g\} \cup \{x_{n+1,t} / 1 \leq t \leq b-1\}$ is a free generators set of $\text{Im}(\alpha)$ and $(B_{i,j})_{1 \leq i < j \leq n}$, $(\rho_{k,l})_{1 \leq k \leq n}$ and $(x_{u,t})_{1 \leq u \leq n}$ are coset representative for the considered generators of $P_n(N_{g,b})$.

There are three types of relations for $P_{n+1}(N_{g,b})$. The first one comes from the relations in $\text{Im}(\alpha)$: there are none here, since this group is free. The second type comes from the relations in $P_n(N_{g,b})$: they lift to the same relations in $P_{n+1}(N_{g,b})$. Finally, the third type arrives by studying the action of $P_n(N_{g,b})$ on $\text{Im}(\alpha)$ by conjugation. We leave to the reader to verify that this action corresponds to the given relations. \square

REMARK 4 What precedes proves that $P_{n+1}(N_{g,b})$ is a semidirect product of the free group

$\pi_1(N_{g,b} \setminus \{z_1, \dots, z_n\}, z_{n+1})$ by $P_n(N_{g,b})$. Therefore, by recurrence, we get that $P_{n+1}(N_{g,b})$ is an iterated semidirect product of (finitely generated) free groups.

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